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INEQUALITIES USING CONVEX COMBINATION CENTERS AND SET BARYCENTERS

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Abstract

The article demonstrates the importance of convex combination centers and set barycenters in creating Jensen's inequalities. Center and barycenter properties are used to the formulation of Jensen's inequality for functions that are convex at one side of its domain.

1. Introduction

Throughout this paper, $I \subseteq \mathbb{R}$ will be an interval with the non-empty interior I^0. Main support for the work will be the famous Jensen's inequalities.

The discrete or basic form (see [3]) says that every convex function $f : I \to \mathbb{R}$ satisfies the inequality

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$$f\left(\sum_{i=1}^{n} p_i x_i\right) \le \sum_{i=1}^{n} p_i f(x_i), \tag{1.1}$$

for all convex combinations $\sum_{i=1}^{n} p_i x_i$ from I.

The continuous or integral form (see [4]) says that every convex μ-integrable function $f : I \to \mathbb{R}$ satisfies the inequality

$$f\left(\frac{1}{\mu(A)}\int_A t d\mu(t)\right) \le \frac{1}{\mu(A)}\int_A f(t) d\mu(t), \tag{1.2}$$

for all μ-measurable subsets $A \subseteq I$ with $\mu(A) > 0$.

2. Convex Combination Centers

The main result in this section is Theorem 2.2 for right convex and right concave functions in the discrete case without restrictions on coefficients.

If $x_i \in I$ are points, and $p_i \in [0, 1]$ are coefficients such that $\sum_{i=1}^{n} p_i = 1$, then the sum $\sum_{i=1}^{n} p_i x_i = c$ belongs to I, and it is called the convex combination from I. The number c itself is called the center of the convex combination. For a continuous function $f : I \to \mathbb{R}$, the convex combination $\sum_{i=1}^{n} p_i f(x_i)$ belongs to $f(I)$. A convex hull of a set X will be denoted by co$X$.

The following theorem is related to convex combinations with common center:

Theorem A. *Let* $x_1, \dots, x_n \in I$ *be points such that* $x_i \notin \mathrm{co}\{x_1, \dots, x_k\}$ *for* $i = k+1, \dots, n$, *where* $1 \le k \le n-1$. *Let* $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ *be non-negative numbers such that* $0 < \sum_{i=1}^{k} \alpha_i = \alpha < \beta = \sum_{i=1}^{n} \alpha_i$.

If one of the equalities

$$\frac{1}{\alpha}\sum_{i=1}^{k}\alpha_i x_i = \frac{1}{\beta}\sum_{i=1}^{n}\alpha_i x_i = \frac{1}{\beta-\alpha}\sum_{i=k+1}^{n}\alpha_i x_i, \tag{2.1}$$

is valid, then the double inequality

$$\frac{1}{\alpha}\sum_{i=1}^{k}\alpha_i f(x_i) \le \frac{1}{\beta}\sum_{i=1}^{n}\alpha_i f(x_i) \le \frac{1}{\beta-\alpha}\sum_{i=k+1}^{n}\alpha_i f(x_i), \tag{2.2}$$

holds for every convex function $f : I \to \mathbb{R}$.

Theorem A was realized in [7, Proposition 2].

Corollary 2.1. *Let* $x_1, \dots, x_n \in I$ *be points such that* $x_i \notin \mathrm{co}\{x_1, \dots, x_k\}$ *for* $i = k+1, \dots, n$, *where* $1 \le k \le n-1$. *Let* $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ *be non-negative numbers such that* $0 < \sum_{i=1}^{k}\alpha_i = \sum_{i=k+1}^{n}\alpha_i$.

If the equality

$$\sum_{i=1}^{k}\alpha_i x_i = \sum_{i=k+1}^{n}\alpha_i x_i, \tag{2.3}$$

is valid, then the inequality

$$\sum_{i=1}^{k}\alpha_i f(x_i) \le \sum_{i=k+1}^{n}\alpha_i f(x_i), \tag{2.4}$$

holds for every convex function $f : I \to \mathbb{R}$.

The next theorem (weighted right convex function theorem) was presented and proved in [2] as the main result.

Theorem B (WRCF-theorem)**.** *Let* $f(u)$ *be a function defined on a real interval I and convex for* $u \ge s \in I$, *and let* $p_1, p_2, \dots, p_n$ *be positive real numbers such that*

$$p = \min\{p_1, p_2, \dots, p_n\}, \quad p_1 + p_2 + \dots + p_n = 1.$$

The inequality

$$p_1f(x_1) + p_2f(x_2) + \ldots + p_nf(x_n) \geq f(p_1x_1 + p_2x_2 + \ldots + p_nx_n),$$

holds for all $x_1, x_2, \ldots, x_n \in I$ *satisfying* $p_1x_1 + p_2x_2 + \ldots + p_nx_n \geq s$ *if and only if*

$$pf(x) + (1-p)f(y) \geq f(s),$$

for all $x, y \in I$ *such that* $x \leq s \leq y$ *and* $px + (1-p)y = s.$

In a similar way, WLCF-theorem (weighted left convex function theorem) and WHCF-theorem (weighted half convex function theorem) were listed in [2]. The main deficiency of these three theorems are convex combinations containing pre-determined coefficients. Especially, the binomial convex combinations include the fixed coefficient $p = \min\{p_1, p_2, \ldots, p_n\}$.

Two subintervals of I specified by the number $s \in I^0$ will be labelled with

$$I_s^{\text{rig}} = \{t \in I | t \geq s\} \quad \text{and} \quad I_s^{\text{lef}} = \{t \in I | t \leq s\}.$$

We formulate the Jensen inequality for right convex and concave functions.

Theorem 2.2 (Discrete case of right convexity and concavity)**.** *Let* $f : I \to \mathbb{R}$ *be a function.*

If f is convex on I_s^{rig} *for some* $s \in I^0$, *then the inequality*

$$f\left(\sum_{i=1}^{n} p_ix_i\right) \leq \sum_{i=1}^{n} p_if(x_i), \tag{2.5}$$

holds for all convex combinations from I satisfying $\sum_{i=1}^{n} p_ix_i \geq s$ *if and only if the inequality*

$$f(px + qy) \leq pf(x) + qf(y), \tag{2.6}$$

holds for all binomial convex combinations from I satisfying $px + qy = s$.

If f is concave on I_s^{rig} for some $s \in I^0$, then the reverse inequalities are valid in (2.5) *and* (2.6).

Proof. Let us prove the sufficiency for the case of right convexity. The proof will be done by induction on the integer n.

The base of induction. Take the binomial convex combination from I such that $px + qy \geq s$. If $px + qy = s$, then the inequality in (2.5) for $n = 2$ holds by the assumption in (2.6). If $px + qy > s$ and $x, y \in I_s^{\text{rig}}$, then we apply the discrete form of Jensen's inequality to get the inequality in (2.5) for $n = 2$. Therefore, suppose $px + qy > s$ with $x < s$ and $p > 0$. Let the point $\overline{x}$ be defined by the equation

$$px + q\overline{x} = s.$$

Then we have $s < px + qy$ and $\overline{x} < y$, and so $s, y \notin \text{co}\{px + qy, \overline{x}\}$. We want to apply Corollary 2.1. First, we solve the equation

$$\alpha_1(px + qy) + \alpha_2\overline{x} = \alpha_3 s + \alpha_4 y, \tag{2.7}$$

with the unknowns $\alpha_1, \alpha_2, \alpha_3,$ and α_4, under the condition $\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4$. Taking $\alpha_1 = 1$, we get $\alpha_2 = q, \alpha_3 = 1,$ and $\alpha_4 = q$. Since the condition in (2.3) is satisfied, it applies

$$f(px + qy) + qf(\overline{x}) \leq f(s) + qf(y), \tag{2.8}$$

by the inequality in (2.4). Since $s = px + q\overline{x}$, it follows $f(s) \leq pf(x) + qf(\overline{x})$ by assumption. After ordering the inequality in (2.8), we get

$$f(px + qy) \leq pf(x) + qf(y).$$

The step of induction. Let $n \ge 2$ be a fixed integer. Suppose that the inequality in (2.5) is true for all corresponding n-membered convex combinations. Let $\sum_{i=1}^{n+1} p_i x_i \ge s$ be a convex combination from I, and without loss of generality, suppose all $p_i > 0$. If all $x_i \ge s$, then the inequality in (2.5) follows from Jensen's inequality for convex functions. Otherwise, if $x_{n+1} \le s$, then

$$\sum_{i=1}^{n} \frac{p_i}{1 - p_{n+1}} x_i \ge s.$$

Using the induction base and premise, it follows:

$$\begin{aligned} f\left(\sum_{i=1}^{n+1} p_i x_i\right) &= f\left(p_{n+1} x_{n+1} + (1 - p_{n+1}) \sum_{i=1}^{n} \frac{p_i}{1 - p_{n+1}} x_i\right) \\ &\le p_{n+1} f(x_{n+1}) + (1 - p_{n+1}) f\left(\sum_{i=1}^{n} \frac{p_i}{1 - p_{n+1}} x_i\right) \\ &\le p_{n+1} f(x_{n+1}) + (1 - p_{n+1}) \sum_{i=1}^{n} \frac{p_i}{1 - p_{n+1}} f(x_i) \\ &= \sum_{i=1}^{n+1} p_i f(x_i), \end{aligned}$$

which ends the proof of the theorem. □

Theorem for left convex and left concave functions can be formulated on the model of Theorem 2.2. We end this section with the formulation of the theorem for half convex and half concave functions.

Theorem 2.3 (Discrete case of half convexity and concavity)**.** *Let* $f : I \to \mathbb{R}$ *be a function.*

If f is convex on I_s^{rig} *or* I_s^{lef} *for some* $s \in I^0$, *then the inequality*

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i), \tag{2.9}$$

holds for all convex combinations from I satisfying $\sum_{i=1}^{n} p_i x_i = s$ *if and only if the inequality*

$$f(px + qy) \leq pf(x) + qf(y), \tag{2.10}$$

holds for all binomial convex combinations from I satisfying $px + qy = s$.

If f *is concave on* I_s^{rig} *or* I_s^{lef} *for some* $s \in I^0$, *then the reverse inequalities are valid in* (2.9) *and* (2.10).

3. Set Barycenters

The main result in this section is Theorem 3.3 for right convex and right concave functions in the integral case.

Integral generalizations of the concept of arithmetic mean in the finite measure spaces are the barycenter of measurable set, and the integral arithmetic mean of integrable function. The basic result on the integral arithmetic means is just the integral form of Jensen's inequality. See [6, pages 44-45].

For a given finite measure μ on I will be assumed that all subintervals of I, and therefore the points, are μ-measurable.

Let $A \subseteq I$ be a μ-measurable set with $\mu(A) > 0$, and 1_A be the identity function on A. If the function 1_A is μ-integrable, then the μ-barycenter of A is defined by

$$\mathcal{B}(A, \mu) = \frac{1}{\mu(A)} \int_A t d\mu(t). \tag{3.1}$$

If a function $f : I \to \mathbb{R}$ is μ-integrable on A, then the μ-arithmetic mean of f on A is defined by

$$\mathcal{M}(f,\ A,\ \mu) = \frac{1}{\mu(A)}\int_A f(t)d\mu(t). \tag{3.2}$$

Note that $\mathcal{M}(1_A,\ A,\ \mu) = \mathcal{B}(A,\ \mu)$. If A is the interval, then its μ-barycenter $\mathcal{B}(A,\ \mu)$ belongs to A. If A is the interval and f is continuous on A, then its μ-arithmetic mean on A belongs to $f(A)$.

Theorem C. *Let* μ *be a finite measure on* I. *Let* $B \subseteq I$ *be a* μ-*measurable set and* $A \subset B$ *be a bounded interval such that* $0 < \mu(A) < \mu(B)$.

If one of the equalities

$$\mathcal{B}(A,\ \mu) = \mathcal{B}(B,\ \mu) = \mathcal{B}(B \setminus A,\ \mu), \tag{3.3}$$

is valid, then the double inequality

$$\mathcal{M}(f,\ A,\ \mu) \leq \mathcal{M}(f,\ B,\ \mu) \leq \mathcal{M}(f,\ B \setminus A,\ \mu), \tag{3.4}$$

holds for every μ-*integrable convex function* $f : I \to \mathbb{R}$.

The version of Theorem C for bounded closed intervals $A = [a,\ b]$ and B was proved in [7, Proposition 1].

A measure μ on I is said to be continuous, if $\mu(\{t\}) = 0$ for every point $t \in I$ (according to the definition in the book [9, page 149]). In the rest of the paper, we will use the continuous finite measure μ on I, which is positive on the intervals, that is, which satisfies $\mu(A) > 0$ for every non-degenerate interval $A \subseteq I$.

If μ is a continuous finite measure on I that is positive on the intervals, then the function

$$x \mapsto \frac{1}{\mu(I_x^{\text{lef}})}\int_{I_x^{\text{lef}}} t d\mu(t) = \mathcal{B}(I_x^{\text{lef}},\ \mu), \tag{3.5}$$

is strictly increasing continuous on I^0. The related function with I_x^{rig} instead of I_x^{lef} has the same properties. We can also use any interval $A \subseteq I$ for the function definition in (3.5) in which case the resulting function is observed on A^0.

Lemma 3.1. *Let μ be a continuous finite measure on I that is positive on the intervals.*

If $a \in I^0$ is a point, then the decreasing series $(A_n)_n$ of bounded intervals $A_n \subset I$ exists so that

$$\mathcal{B}(A_n, \mu) = a \quad \textit{and} \quad \bigcap_{n=1}^{\infty} A_n = \{a\}.$$

Proof. Take a point $a \in I^0$. Let us show the basic and the iterative step.

In the first step, we choose the points $x_1, y_1 \in I$ such that $x_1 < a < y_1$, and determine the μ-barycenter of the interval $[x_1, y_1]$:

$$a_1 = \frac{1}{\mu([x_1, y_1])} \int_{[x_1, y_1]} t d\mu(t).$$

If $a_1 = a$, then we take $A_1 = [x_1, y_1]$. If $a_1 > a$, then we observe the function $g : [a, y_1] \to \mathbb{R}$ defined by

$$g(x) = \frac{1}{\mu([x_1, x])} \int_{[x_1, x]} t d\mu(t) - a.$$

Since g is continuous, $g(a) < 0$ and $g(y_1) > 0$, there must be a point $\overline{y}_1 \in < a, y_1 >$ such that $g(\overline{y}_1) = 0$. In this case, we can take $A_1 = [x_1, \overline{y}_1]$. If $a_1 < a$, then we increase x_1 until we obtain one of the previous two cases.

In the next step, if $A_1 = [x_1, y_1]$, we take the points

$$x_2 = \frac{x_1 + a}{2} \quad \text{and} \quad y_2 = \frac{a + y_1}{2},$$

and repeat the previous procedure to determine A_2. □

Lemma 3.2. *Let* μ *be a continuous finite measure on I that is positive on the intervals. Let* $B \subseteq I$ *be an interval with the* μ*-barycenter* $b = \mathcal{B}(B, \mu)$.

If $a \in B^0$ *is a point different from b, then the interval* $A \subset B$ *exists so that the binomial convex combination*

$$b = \frac{\mu(A)}{\mu(B)} a + \frac{\mu(B \setminus A)}{\mu(B)} \overline{a}, \tag{3.6}$$

holds with $a = \mathcal{B}(A, \mu)$ *and* $\overline{a} = \mathcal{B}(B \setminus A, \mu)$.

Proof. In the case $a < b$, we use the function

$$g(x) = \frac{1}{\mu(B_x^{\text{lef}})} \int_{B_x^{\text{lef}}} t d\mu(t) - a, \tag{3.7}$$

on the domain B^0. The function g is strictly increasing continuous on B^0, and has the unique zero-point x_0. Taking $A = B_{x_0}^{\text{lef}}$, the equality in (3.6) is evident. Also, $B \setminus A = B_{x_0}^{\text{rig}} \setminus \{x_0\} \subset B_a^{\text{rig}}$.

In the case $a > b$, we use the function g with the sets B_x^{rig} instead of the sets B_x^{lef}, and take $A = B_{x_0}^{\text{rig}}$. In this case, $B \setminus A = B_{x_0}^{\text{lef}} \setminus \{x_0\} \subset B_a^{\text{lef}}$.

□

Theorem 3.3 (Integral case of right convexity and concavity)**.** *Let* μ *be a continuous finite measure on I that is positive on the intervals, and* $f : I \to \mathbb{R}$ *be a* μ*-integrable function.*

If f is convex on I_s^{rig} *for some* $s \in I^0$, *then the inequality*

$$f\left(\frac{1}{\mu(B)}\int_B t d\mu(t)\right) \leq \frac{1}{\mu(B)}\int_B f(t) d\mu(t), \tag{3.8}$$

holds for all intervals $B \subseteq I$ *satisfying*

$$\frac{1}{\mu(B)}\int_B t d\mu(t) \geq s, \tag{3.9}$$

if and only if the inequality

$$f\left(\frac{1}{\mu(A)}\int_A t d\mu(t)\right) \leq \frac{1}{\mu(A)}\int_A f(t) d\mu(t), \tag{3.10}$$

holds for all bounded intervals $A \subseteq I$ *satisfying*

$$\frac{1}{\mu(A)}\int_A t d\mu(t) = s. \tag{3.11}$$

If f is concave on I_s^{rig} *for some* $s \in I^0$, *then the reverse inequalities are valid in* (3.8) *and* (3.10).

Proof. Let us prove the sufficiency for the case of right convexity. Suppose B is an interval from I such that its μ-barycenter

$$b = \frac{1}{\mu(B)}\int_B t d\mu(t) \geq s.$$

If $s \notin B^0$ in which case $B \subseteq I_s^{\text{rig}}$, the inequality in (3.8) follows from the integral form of Jensen's inequality for convex functions.

Suppose $s \in B^0$. If $s = b$, then using Lemma 3.1, we can determine the bounded interval $A \subset B$ with the μ-barycenter s. Using the assumption in (3.10), and the inequality in (3.4) under the condition $\mathcal{B}(A, \mu) = \mathcal{B}(B, \mu)$, we get

$$f\left(\frac{1}{\mu(B)}\int_B t d\mu(t)\right) = f\left(\frac{1}{\mu(A)}\int_A t d\mu(t)\right) \leq \frac{1}{\mu(A)}\int_A f(t) d\mu(t)$$

$$\leq \frac{1}{\mu(B)}\int_B f(t) d\mu(t).$$

If $s < b$, then using Lemma 3.2, we can determine the interval $S \subset B$ so that the binomial convex combination

$$b = \frac{\mu(S)}{\mu(B)} s + \frac{\mu(B \setminus S)}{\mu(B)} \bar{s}, \tag{3.12}$$

holds with $s = \mathcal{B}(S, \mu)$ and $\bar{s} = \mathcal{B}(B \setminus S, \mu)$. Since $s, \bar{s} \in I_s^{\text{rig}}$, we can apply the discrete form of Jensen's inequality to the equality in (3.12), and get

$$f(b) \leq \frac{\mu(S)}{\mu(B)} f(s) + \frac{\mu(B \setminus S)}{\mu(B)} f(\bar{s}). \tag{3.13}$$

Since $B \setminus S \subset B_s^{\text{rig}} \subseteq I_s^{\text{rig}}$ is also valid, we may apply the integral form of Jensen's inequality to the barycenter $\bar{s}$, and have the estimate for $f(\bar{s})$:

$$f(\bar{s}) = f\left(\frac{1}{\mu(B \setminus S)}\int_{B \setminus S} t d\mu(t)\right) \leq \frac{1}{\mu(B \setminus S)}\int_{B \setminus S} f(t) d\mu(t).$$

Now, we include Lemma 3.1 to determine the bounded interval $A \subset S$ with the μ-barycenter s. Using the assumption in (3.10), and the inequality in (3.4) under the condition $\mathcal{B}(A, \mu) = \mathcal{B}(S, \mu)$, we obtain the estimate for $f(s)$:

$$f(s) = f\left(\frac{1}{\mu(A)}\int_A t d\mu(t)\right) \leq \frac{1}{\mu(A)}\int_A f(t) d\mu(t)$$

$$\leq \frac{1}{\mu(S)}\int_S f(t) d\mu(t).$$

After ordering the inequality in (3.13), it follows:

$$f\left(\frac{1}{\mu(B)}\int_B td\mu(t)\right) \leq \frac{1}{\mu(B)}\int_S f(t)d\mu(t) + \frac{1}{\mu(B)}\int_{B\setminus S} f(t)d\mu(t)$$

$$= \frac{1}{\mu(B)}\int_S f(t)d\mu(t),$$

and the proof is done. □

Using the previous theorem, we can express the theorem for left convex and left concave functions in the integral case. Combining these cases, we get the theorem for half convex and half concave functions in the integral case.

Theorem 3.4 (Integral case of half convexity and concavity)**.** *Let* μ *be a continuous finite measure on* I *that is positive on the intervals, and* $f : I \to \mathbb{R}$ *be a* μ*-integrable function.*

If f is convex on I_s^{rig} *or* I_s^{lef} *for some* $s \in I^0$, *then the inequality*

$$f\left(\frac{1}{\mu(B)}\int_B td\mu(t)\right) \leq \frac{1}{\mu(B)}\int_B f(t)d\mu(t), \tag{3.14}$$

holds for all intervals $B \subseteq I$ *satisfying*

$$\frac{1}{\mu(B)}\int_B td\mu(t) = s, \tag{3.15}$$

if and only if the inequality

$$f\left(\frac{1}{\mu(A)}\int_A td\mu(t)\right) \leq \frac{1}{\mu(A)}\int_A f(t)d\mu(t), \tag{3.16}$$

holds for all bounded intervals $A \subseteq I$ *satisfying*

$$\frac{1}{\mu(A)}\int_A td\mu(t) = s. \tag{3.17}$$

If f is concave on I_s^{rig} or I_s^{lef} for some $s \in I^0$, then the reverse inequalities are valid in (3.14) *and* (3.16).

4. Applications to Quasi-Arithmetic Means

In the applications of convexity to quasi-arithmetic means, we use strictly monotone continuous functions $\varphi, \psi : I \to \mathbb{R}$ such that ψ is convex with respect to $\varphi(\psi$ is φ-convex$)$, that is, $f = \psi \circ \varphi^{-1}$ is convex (according to the terminology in the book [8, Definition 1.19]). A similar notation is used for concavity. Very general forms of discrete and integral quasi-arithmetic means, and its refinements, were studied in [5]. A good approach to means can be seen in [1].

4.1. Discrete case

Let $\sum_{i=1}^{n} p_i x_i$ be a convex combination from I. The discrete φ-quasi-arithmetic mean of the points x_i with the coefficients p_i is the number

$$\mathcal{M}_{\varphi}(x_i;\, p_i) = \varphi^{-1}\left(\sum_{i=1}^{n} p_i \varphi(x_i)\right), \tag{4.1}$$

which belongs to I, because the convex combination $\sum_{i=1}^{n} p_i \varphi(x_i)$ belongs to $\varphi(I)$. If φ is the identity function on I, that is, $\varphi = 1_I$, then the discrete 1_I-quasi-arithmetic mean is just the convex combination $\sum_{i=1}^{n} p_i x_i$.

The following is the theorem for right convexity and right concavity for quasi-arithmetic means in the discrete case:

Theorem 4.1 (Discrete case of right convexity and concavity for quasi-arithmetic means). *Let $\varphi, \psi : I \to \mathbb{R}$ be strictly monotone continuous functions, and $J = \varphi(I)$.*

If ψ *is either* φ*-convex on* $J_{\varphi(s)}^{\mathrm{rig}}$ *for some* $\varphi(s) \in J^0$ *and increasing on I or* φ*-concave on* $J_{\varphi(s)}^{\mathrm{rig}}$ *and decreasing on I, then the inequality*

$$\mathcal{M}_{\varphi}(x_i;\ p_i) \leq \mathcal{M}_{\psi}(x_i;\ p_i), \tag{4.2}$$

holds for all convex combinations from J satisfying $\sum_{i=1}^{n} p_i \varphi(x_i) \geq \varphi(s)$ *if and only if the inequality*

$$\mathcal{M}_{\varphi}(x,\ y;\ p,\ q) \leq \mathcal{M}_{\psi}(x,\ y;\ p,\ q), \tag{4.3}$$

holds for all binomial convex combinations from J satisfying $p\varphi(x) + q\varphi(y) = \varphi(s)$.

If ψ *is either* φ*-convex on* $J_{\varphi(s)}^{\mathrm{rig}}$ *for some* $\varphi(s) \in J^0$ *and decreasing on I or* φ*-concave on* $J_{\varphi(s)}^{\mathrm{rig}}$ *and increasing on I, then the reverse inequalities are valid in* (4.2) *and* (4.3).

Proof. We prove the case in which the function ψ is φ-convex on the interval $J_{\varphi(s)}^{\mathrm{rig}}$ and increasing on the interval I. Put $f = \psi \circ \varphi^{-1}$.

First, if we apply DRCF-theorem to the function $f : J \to \mathbb{R}$ convex on the interval $J_{\varphi(s)}^{\mathrm{rig}}$, then we have

$$f\left(\sum_{i=1}^{n} p_i \varphi(x_i)\right) \leq \sum_{i=1}^{n} p_i f(\varphi(x_i)),$$

holds for all convex combinations from J satisfying $\sum_{i=1}^{n} p_i \varphi(x_i) \geq \varphi(s)$ if and only if

$$f(p\varphi(x) + q\varphi(y)) \leq pf(\varphi(x)) + qf(\varphi(y)),$$

holds for all binomial convex combinations from J satisfying $p\varphi(x) + q\varphi(y) = \varphi(s)$.

Second, after applying the increasing function ψ^{-1} on the above inequalities, it follows:

$$\mathcal{M}_{\varphi}(x_i;\ p_i) = \varphi^{-1}\left(\sum_{i=1}^{n} p_i\varphi(x_i)\right) \le \psi^{-1}\left(\sum_{i=1}^{n} p_i\psi(x_i)\right) = \mathcal{M}_{\psi}(x_i;\ p_i),$$

holds for all binomial convex combinations from J satisfying $\sum_{i=1}^{n} p_i\varphi(x_i) \ge \varphi(s)$ if and only if

$$\mathcal{M}_{\varphi}(x,\ y;\ p,\ q) = \varphi^{-1}(p\varphi(x) + q\varphi(y)) \le \psi^{-1}(p\varphi(x) + q\varphi(y)) = \mathcal{M}_{\psi}(x,\ y;\ p,\ q),$$

holds for all $x,\ y \in I$ satisfying $p\varphi(x) + q\varphi(y) = \varphi(s)$. □

4.2. Integral case

Let $A \subseteq I$ be a μ-measurable set with $\mu(A) > 0$, and $\varphi : I \to \mathbb{R}$ be a strictly monotone continuous function that is μ-integrable on A. The integral φ-quasi-arithmetic mean of the set A with respect to the measure μ is the point

$$\mathcal{M}_{\varphi}(A,\ \mu) = \varphi^{-1}\left(\frac{1}{\mu(A)}\int_A \varphi(t)d\mu(t)\right). \tag{4.4}$$

If A is the interval, then its φ-quasi-arithmetic mean $\mathcal{M}_{\varphi}(A,\ \mu)$ belongs to A because the point $\frac{1}{\mu(A)}\int_A \varphi(t)d\mu(t)$ belongs to $\varphi(A)$. If A is not connected, then $\mathcal{M}_{\varphi}(A,\ \mu)$ may be outside of A. If $\varphi = 1_I$, then the integral 1_I-quasi-arithmetic mean of A is just the barycenter $\mathcal{B}(A,\ \mu)$.

Theorem for right convexity and right concavity for quasi-arithmetic means in the integral case as follows:

Theorem 4.2 (Integral case of right convexity and concavity for quasi-arithmetic means)**.** *Let* μ *be a continuous finite measure on I that is* *positive on the intervals. Let* $\varphi, \psi : I \to \mathbb{R}$ *be* μ*-integrable strictly monotone continuous functions, and* $J = \varphi(I)$.

If ψ *is either* φ*-convex on* $J^{\mathrm{rig}}_{\varphi(s)}$ *for some* $\varphi(s) \in J^0$ *and increasing on* I *or* φ*-concave on* $J^{\mathrm{rig}}_{\varphi(s)}$ *and decreasing on* I, *then the inequality*

$$\mathcal{M}_{\varphi}(B, \mu) \le \mathcal{M}_{\psi}(B, \mu), \tag{4.5}$$

holds for all intervals $B \subseteq I$ *satisfying*

$$\frac{1}{\mu(B)} \int_B \varphi(t) d\mu(t) \ge \varphi(s), \tag{4.6}$$

if and only if the inequality

$$\mathcal{M}_{\varphi}(A, \mu) \le \mathcal{M}_{\psi}(A, \mu), \tag{4.7}$$

holds for all bounded intervals $A \subseteq I$ *satisfying*

$$\frac{1}{\mu(A)} \int_A \varphi(t) d\mu(t) = \varphi(s). \tag{4.8}$$

If ψ *is either* φ*-convex on* $J^{\mathrm{rig}}_{\varphi(s)}$ *for some* $\varphi(s) \in J^0$ *and decreasing on* I *or* φ*-concave on* $J^{\mathrm{rig}}_{\varphi(s)}$ *and increasing on* I, *then the reverse inequalities are valid in* (4.5) *and* (4.7).

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